Partial Differential Equations: Midterm Exam

Aletta Jacobshal 01, Friday 4 March 2016, 14:00 - 16:00
Duration: 2 hours

- Solutions should be complete and clearly present your reasoning.
- 10 points are "free". There are 4 questions and the total number of points is 100. The midterm grade is the total number of points divided by 10.
- Do not forget to very clearly write your **full name** and **student number** on the envelope.
- Do not seal the envelope.

Question 1 (20 points)

Consider the first order partial differential equation

$$y u_x + e^x u_y = 0, \tag{1}$$

where $u = u(x, y)$.

(a) (14 points) Find the general solution of Eq. (1).

Solution

We consider the ordinary differential equation

$$\frac{dy}{dx} = \frac{e^x}{y},$$

which gives the characteristic curves. This equation is separable and it gives

$$y\,dy = e^x\,dx.$$

The integration gives

$$\tfrac{1}{2}y^2 = e^x + C.$$

Solving for the integration constant C we find

$$C = \tfrac{1}{2}y^2 - e^x.$$

Therefore the general solution is

$$u(x, y) = f(\tfrac{1}{2}y^2 - e^x)$$

where f is an arbitrary function.

(b) (6 points) Find the solution of Eq. (1) with the auxiliary condition $u(0, y) = y^4$.

Solution

We have

$$u(0, y) = f(\tfrac{1}{2}y^2 - 1) = y^4.$$

Setting $\frac{1}{2}y^2 - 1 = z$ we get $y^2 = 2z + 2$ and therefore $f(z) = 4(z+1)^2 = (2z+2)^2$. Therefore the solution that satisfies the given auxiliary condition is

$$u(x, y) = \left(y^2 - 2e^x + 2\right)^2.$$

Question 2 (25 points)

Consider the Schrödinger equation $u_t = iku_{xx}$ for real k in the interval $0 < x < \ell$ with the boundary conditions $u_x(0,t) = 0$ and $u(\ell,t) = 0$.

(a) (7 points) Show that if we consider separated solutions of the form $u(x,t) = X(x)T(t)$ then we get the differential equations $-X'' = \lambda X$ and $T' = -ik\lambda T$. What are the boundary conditions satisfied by $X(x)$?

Solution

We substitute $u(x,t) = X(x)T(t)$ into the Schrödinger equation and we find

$$XT' = ikX''T.$$

Dividing both sides by $ikXT$ we get

$$\frac{T'}{ikT} = \frac{X''}{X} = -\lambda,$$

where λ is constant. Therefore we find the given equations

$$-X'' = \lambda X, \quad T' = -ik\lambda T.$$

The boundary conditions for $X(x)$ are $X'(0) = 0$ and $X(\ell) = 0$.

(b) (12 points) It is given that the eigenvalues are $\lambda_n = \beta_n^2 > 0$ where

$$\beta_n = \frac{\pi}{\ell}\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, 3, \ldots.$$

Solve the differential equations for $X(x)$ and $T(t)$ for the given eigenvalues and write the general solution for $u(x,t)$.

Solution

We have

$$-X_n'' = \beta_n^2 X_n$$

with the solution

$$X_n = C_n \cos(\beta_n x) + D_n \sin(\beta_n x),$$

which gives

$$X_n' = -\beta_n C_n \sin(\beta_n x) + \beta_n D_n \cos(\beta_n x).$$

Then

$$X_n'(0) = \beta_n D_n = 0,$$

which gives $D_n = 0$. Therefore

$$X_n = C_n \cos(\beta_n x),$$

and we can check that $X_n(\ell) = C_n \cos((n + 1/2)\pi) = 0$. Furthermore, we can set $C_n = 1$ since eigenfunctions are determined up to a constant factor.

Then we have

$$T_n' = -ik\beta_n^2 T_n,$$

with solution

$$T_n = A_n \exp(-ik\beta_n^2 t).$$

Therefore the general solution can be written as

$$u(x,t) = \sum_{n=0}^{\infty} A_n \exp(-ik\beta_n^2 t)\cos(\beta_n x).$$

(c) (6 points) Find the solution $u(x,t)$ if it satisfies the initial condition

$$u(x,0) = \frac{1}{2}\cos\left(\frac{3\pi x}{2\ell}\right) - \frac{3}{8}\cos\left(\frac{7\pi x}{2\ell}\right).$$

Solution
Setting $t = 0$ in the general solution we get

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos(\beta_n x).$$

Comparing with the given $u(x,0)$ which includes the terms with $n = 1$ and $n = 3$ we conclude that $A_1 = 1/2$, $A_3 = -3/8$, and $A_n = 0$ in all other cases. The corresponding β_n are $\beta_1 = 3\pi/2\ell$ and $\beta_3 = 7\pi/2\ell$. Therefore the solution is

$$u(x,t) = \frac{1}{2}\exp\left(-\frac{9ik\pi^2 t}{4\ell^2}\right)\cos\left(\frac{3\pi x}{2\ell}\right) - \frac{3}{8}\exp\left(-\frac{49ik\pi^2 t}{4\ell^2}\right)\cos\left(\frac{7\pi x}{2\ell}\right).$$

Question 3 (20 points)

Consider the second order partial differential equation

$$2u_{xx} - u_{xy} - u_{yy} = 0. \qquad (2)$$

where $u = u(x,y)$.

(a) (5 points) Classify the partial differential equation (2) as elliptic, hyperbolic, or parabolic.

Solution
We have $a_{11} = 2$, $a_{12} = -1/2$, $a_{22} = -1$, so

$$a_{12}^2 = \frac{1}{4} > -2 = a_{11}a_{22}.$$

Therefore the equation is hyperbolic.

(b) (15 points) Find a linear coordinate transformation $(x,y) \to (s,t)$ such that Eq. (2) reduces to the form $u_{st} = 0$; express x and y in terms of s and t. Hint: factorize the second order operator corresponding to the given equation as the product of two first order operators.

Solution
The easiest way is to factorize the linear operator

$$\mathcal{L} = 2\partial_x^2 - \partial_x\partial_y - \partial_y^2 = (2\partial_x + \partial_y)(\partial_x - \partial_y).$$

Then define

$$\partial_s = 2\partial_x + \partial_y, \quad \partial_t = \partial_x - \partial_y,$$

and in matrix form

$$\begin{pmatrix} \partial_s \\ \partial_t \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix},$$

or

$$x = 2s + t, \quad y = s - t.$$

Question 4 (25 points)

Consider the eigenvalue problem $-X'' = \lambda X$ for $-\pi < x < \pi$ with boundary conditions $X(-\pi) = -X(\pi)$ and $X'(-\pi) = -X'(\pi)$.

(a) (6 points) Prove that $\lambda = 0$ is not an eigenvalue.

Solution

For $\lambda = 0$ we have the equation $X'' = 0$ with solution $X = Cx + D$. Then $X'(x) = C$. The equation $X'(-\pi) = -X'(\pi)$ gives $C = -C$ and therefore $C = 0$. Moreover the equation $X(-\pi) = -X(\pi)$ gives $D = -D$ so $D = 0$. Therefore we find that $X = 0$ which is not possible.

(b) (7 points) Prove that there are no negative eigenvalues.

Solution

For $\lambda = -\gamma^2 < 0$ we have the solution

$$X(x) = C\cosh(\gamma x) + D\sinh(\gamma x).$$

Then

$$X'(x) = \gamma C\sinh(\gamma x) + \gamma D\cosh(\gamma x).$$

The equation $X(-\pi) = -X(\pi)$ gives

$$C\cosh(-\gamma\pi) + D\sinh(-\gamma\pi) = -C\cosh(\gamma\pi) - D\sinh(\gamma\pi) \Leftrightarrow C\cosh(\gamma\pi) = 0 \Leftrightarrow C = 0.$$

The equation $X'(-\pi) = -X'(\pi)$ gives

$$-\gamma C\sinh(-\gamma\pi) + \gamma D\cosh(-\gamma\pi) = \gamma C\sinh(\gamma\pi) - \gamma D\cosh(-\gamma\pi) \Leftrightarrow D\cosh(\gamma\pi) = 0 \Leftrightarrow D = 0.$$

Therefore we find that $X = 0$ which is not possible.

(c) (12 points) Compute the positive eigenvalues and the corresponding eigenfunctions for this problem.

Solution
For $\lambda = \beta^2 > 0$ we have the solution

$$X(x) = C\cos(\beta x) + D\sin(\beta x).$$

Then

$$X'(x) = -\beta C\sin(\beta x) + \beta D\cos(\beta x).$$

The equation $X(-\pi) = -X(\pi)$ gives

$$C\cos(-\beta\pi) + D\sin(-\beta\pi) = -C\cos(\beta\pi) - D\sin(\beta\pi) \Leftrightarrow C\cos(\beta\pi) = 0.$$

The equation $X'(-\pi) = -X'(\pi)$ gives

$$-\beta C\sin(-\beta\pi) + \beta D\cos(-\beta\pi) = \beta C\sin(\beta\pi) - \beta D\cos(-\beta\pi) \Leftrightarrow D\cos(\beta\pi) = 0.$$

Therefore, $\cos(\beta\pi) = 0$ which gives

$$\beta_n = n - \frac{1}{2}, \quad n = 1, 2, \ldots.$$

Therefore the eigenvalues are

$$\lambda_n = \left(n - \frac{1}{2}\right)^2, \quad n = 1, 2, \ldots.$$

The corresponding eigenfunctions are

$$X_n = C_n\cos(\beta_n x) + D_n\sin(\beta_n x),$$

where C_n, D_n are arbitrary constants with the only restriction that $C_n^2 + D_n^2 \neq 0$, that is, they are not simultaneously zero.

End of the exam (Total: 90 points)